Notes on Measures and Integrals

Integrable vector functions

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In this article, $X = (X, \mathcal{S}, \mu)$ denotes a measure space; E denotes a separable Banach space; $\mathcal{F}_E = \mathcal{F}_E(X)$ denotes the collection of all E-valued, relatively measurable functions on X having measurable supports; $\mathcal{F}_+ = \mathcal{F}_+(X)$ denotes the collection of all non-negative, extended real-valued, relatively measurable functions on X having measurable supports; 'SF' is short for 'simple function', 'ISF' for 'integrable simple function'.

1 Integrable functions

1.1 Lemma. Let (f_n) be a sequence of functions in $\mathcal{F}_E$ converging a.e. to a function $f \in \mathcal{F}_E$. If there exists an integrable function $g \in \mathcal{F}_+$ such that $|f_n| \leqslant g$ for all n, then $\lim_n \int |f - f_n|\, d\mu = 0$.

Note. This lemma contains the essence of Lebesgue's dominated convergence theorem.

Proof The integrability of g implies that there exists a σ-finite support A for g. It also implies that, given any $\varepsilon > 0$, there exists a measurable subset A_1 of A such that $\mu(A_1) < \infty$ and

$$\int_{A-A_1} g\, d\mu < \varepsilon. \tag{1}$$

Since $|f_n|, |f| \leqslant g$ a.e., (1) implies

$$\int_{A-A_1} |f - f_n| \, d\mu < 2\varepsilon. \tag{2}$$

Furthermore, the indefinite integral of g is absolutely continuous in the sense that there exists $\delta > 0$ such that $\int_B g \, d\mu < \varepsilon$ whenever $\mu(B) < \delta$.

Now, by Egorov's theorem, we can find a measurable subset A_2 of A_1 such that $\mu(A_1 - A_2) < \delta$ and $f_n \to f$ uniformly on A_2. The latter condition implies that $|f - f_n| < \frac{\varepsilon}{\mu(A_2)}$ holds on A_2 for all $n \geqslant N$ if N is chosen sufficiently large. Hence it follows that

$$\int_{A_1-A_2} |f - f_n| \, d\mu < 2\varepsilon, \tag{3}$$

and

$$\int_{A_2} |f - f_n| \, d\mu < \varepsilon \quad (n \geqslant N). \tag{4}$$

Combining (2), (3) and (4) together, we obtain

$$\int |f - f_n| \, d\mu < 5\varepsilon \quad (n \geqslant N).$$

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1.2 Definition. A function $f \in \mathcal{F}_E$ is said to be *integrable* iff $\int |f| \, d\mu < \infty$.

Note. $f \in \mathcal{F}_E$ implies $|f| \in \mathcal{F}_+$.

1.3 Theorem. Let $f \in \mathcal{F}_E$ be an integrable function. Then, there exists a sequence (s_n) of E-valued ISFs such that

$$\lim_n \int |f - s_n| \, d\mu = 0. \tag{5}$$

Note. We say (s_n) *mean converges* to f if (5) is satisfied.

Proof Because f has a σ-finite support, there exists a sequence (s_n) of E-valued ISFs such that $s_n \to f$ a.e. and $|s_n| \leqslant f$. Therefore, (5) is an immediate consequence of Lemma 1.1. ||

1.4 Definition. If $f \in \mathcal{F}_E$ is integrable, and if (s_n) is any sequence of E-valued ISFs mean convergent to f, we define the *integral* of f to be the value

$$\int f \, d\mu = \lim_n \int s_n \, d\mu. \tag{6}$$

Note. Because

$$\left| \int s_m \, d\mu - \int s_n \, d\mu \right| \leqslant \int |\, s_m - s_n \,| \, d\mu \leqslant \int |\, s_m - f \,| \, d\mu + \int |\, f - s_n \,| \, d\mu,$$

(5) implies that the sequence $(\int s_n \, d\mu)$ of values in E is fundamental, so that it is convergent in E. That the limit (6) is independent of the choice of (s_n) is not difficult to see. Hence the integral of f is well-defined by (6).

Note. If f is non-negative real-valued, the integral $\int f \, d\mu$ has been defined as the sup of $\int s \, d\mu$, where s is a variable non-negative real SF such that $s \leqslant f$. This $\int f \, d\mu$ coincides with our current definition, as proved in the article "Integration of non-negative functions".

2 Properties of integrable functions

2.1 Theorem. Let $f \in \mathcal{F}_E$. If f is integrable, then $|\, f \,|$ is integrable, and

$$\left| \int f \, d\mu \right| \leqslant \int |\, f \,| \, d\mu. \tag{7}$$

Proof The integrability of $|\, f \,|$ is trivial from our definition. Let (s_n) be a sequence of ISFs mean converging to f. Since

$$|\, |\, f \,| - |\, s_n \,| \,| \leqslant |\, f - s_n \,| \quad (n = 1, 2, \cdots),$$

it follows that the sequence $(|\, s_n \,|)$ mean converges to $|\, f \,|$, and hence that $\int |\, f \,| \, d\mu = \lim_n \int |\, s_n \,| \, d\mu$. Thus, the inequality (7) is obtained by taking limits from both sides of $\left| \int s_n \, d\mu \right| \leqslant \int |\, s_n \,| \, d\mu$. ||

2.2 Theorem. Let $f, g \in \mathcal{F}_E$. If f is integrable and $f = g$ a.e., then g is integrable and $\int f \, d\mu = \int g \, d\mu$.

Proof $f = g$ a.e. implies $|f| = |g|$ a.e., so that $\int |f| \, d\mu = \int |g| \, d\mu$, and hence g is integrable.

If (s_n) is a sequence of ISFs mean converging to f, then it also mean converges to g, and therefore both integrals coincide. ||

2.3 Theorem. Let $f, g \in \mathcal{F}_E$ and c be a scalar. If f and g are integrable, then $f + cg$ is integrable and

$$\int (f + cg) \, d\mu = \int f \, d\mu + c \int g \, d\mu. \tag{8}$$

Proof If (s_n) and (t_n) are sequences of ISFs mean converging to f and g, respectively, then the sequence $(s_n + ct_n)$ is mean convergent to $f + cg$, whence

$$\int (f + cg) \, d\mu = \lim_n \int (s_n + ct_n) \, d\mu = \lim_n \int s_n \, d\mu + c \lim_n \int t_n \, d\mu$$
$$= \int f \, d\mu + c \int g \, d\mu.$$

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2.4 Theorem (Lebesgue's dominated convergence theorem). Let (f_n) be a sequence of functions in $\mathcal{F}_E$ converging a.e. to a function $f \in \mathcal{F}_E$. If there exists an integrable function $g \in \mathcal{F}_+$ such that $|f_n| \leqslant g$ for all n, then

$$\int f \, d\mu = \lim_n \int f_n \, d\mu. \tag{9}$$

Proof Note that $|f_n| \leqslant g$ a.e. implies that f_n is integrable, so that $\int f_n \, d\mu$ makes sense. The same for $\int f \, d\mu$. Because the inequality

$$\left| \int f \, d\mu - \int f_n \, d\mu \right| \leqslant \int |f - f_n| \, d\mu \tag{10}$$

holds for all n, (9) is an immediate consequence from Lemma 1.1. ||

Note. To deduce (10), we use the linearity (8) of the integral and the inequality (7).

3 Indefinite integrals

3.1 Definition. Let $f \in \mathcal{F}_E$ and $A \in \mathcal{S}$. Suppose f is integrable. Since $|\chi_A f| \leqslant |f|$, it follows that $\chi_A f$ is integrable. We denote

$$\int_A f\,d\mu = \int \chi_A f\,d\mu,$$

and call it the *integral of f over A*.

Note. If A is a support for f, then $\chi_A f = f$, so that

$$\int_A f\,d\mu = \int f\,d\mu.$$

3.2 Theorem. Given any integrable function $f \in \mathcal{F}_E$, the E-valued set function ν defined by

$$\nu(A) = \int_A f\,d\mu$$

is a E-valued vector measure.

Proof Let $A, B \in \mathcal{S}$ and $A \perp B$. Then

$$\nu(A \cup B) = \int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu = \nu(A) + \nu(B). \qquad (11)$$

This is obvious from the identity $\chi_{A\cup B} f = \chi_A f + \chi_B f$ and the additivity of the integral. (11) shows that ν is finitely additive.

Next, let $A_n \in \mathcal{S}$, $A_n \uparrow$ $(n = 1, 2, \cdots)$ and $A = \lim_n A_n$. Then

$$\nu(A) = \int_A f\,d\mu = \lim_n \int_{A_n} f\,d\mu = \lim_n \nu(A_n). \qquad (12)$$

For, $\chi_A f = \lim_n \chi_{A_n} f$ everywhere, and since f is integrable, this together with the inequality $|\chi_{A_n} f| \leqslant |f|$ implies $\int \chi_A f\,d\mu = \lim_n \int \chi_{A_n} f\,d\mu$ by the convergence theorem 2.4. Hence (12) is proved.

From the finite additivity of ν and (12), it follows that ν is σ-additive, i.e. ν is a vector measure. ||

Note. For the theory of vector measures, see my another article "Vector measures".

3.3 Definition. The vector measure ν defined as above is called the *indefinite integral* of the integrable function f.

3.4 Theorem. If ν is the indefinite integral of an integrable function $f \in \mathcal{F}_E$, then ν is absolutely continuous, i.e. given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(A)| < \varepsilon$ whenever $A \in \mathcal{S}$, $\mu(A) < \delta$.

Proof In view of the fact that any indefinite integral of integrable non-negative function is absolutely continuous, the statement is trivial because

$$|\nu(A)| = \left| \int_A f \, d\mu \right| \leqslant \int_A |f| \, d\mu.$$

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4 Sequences of integrable functions

4.1 Definition. Let (f_n) be a sequence of integrable functions. (f_n) is said to *mean converge* to 0 iff

$$\lim_{n \to \infty} \int |f_n| \, d\mu = 0.$$

4.2 Definition. Let (f_n) be a sequence of integrable functions. (f_n) is said to be *mean fundamental* iff

$$\lim_{m,n \to \infty} \int |f_m - f_n| \, d\mu = 0.$$

4.3 Theorem. If (f_n), (g_n) are mean fundamental sequences of integrable functions and c is a scalar, then the sequence $(f_n + c g_n)$ is mean fundamental.

Proof Trivial. ||

4.4 Theorem. If (f_n) is a mean fundamental sequence of integrable functions, then the sequence $(|f_n|)$ is mean fundamental.

Proof Trivial. ||

4.5 Theorem. Let (f_n) be a mean fundamental sequence of integrable functions and let ν_n be the indefinite integral of f_n for each $n = 1, 2, \cdots$. Then, the sequence (ν_n) of vector measures is uniformly fundamental.

Proof This is obvious from the following inequality:

$$|\,\nu_m(A) - \nu_n(A)\,| = \left| \int_A (f_m - f_n)\, d\mu \right| \leqslant$$
$$\leqslant \int_A |\, f_m - f_n \,|\, d\mu \leqslant \int |\, f_m - f_n \,|\, d\mu < \varepsilon,$$

which holds for all $A \in \mathcal{S}$ and all $m, n \geqslant n_0$. ||

4.6 Theorem. Let (h_n) be a sequence of integrable functions. Then (h_n) mean converges to 0 if and only if the following are satisfied:

1. (h_n) is mean fundamental.
2. (h_n) converges to 0 in measure.

Proof ('only if') Mean fundamentality is obvious from the inequality: $|\,h_m - h_n\,| \leqslant |\,h_m\,| + |\,h_n\,|$.

Now, given any $\varepsilon > 0$, let

$$A_n = \left\{ x \in X \;\middle|\; |\,h_n(x)\,| \geqslant \varepsilon \right\}. \tag{13}$$

Then

$$\int |\,h_n\,|\; d\mu \geqslant \int_{A_n} |\,h_n\,|\; d\mu \geqslant \varepsilon\mu(A_n),$$

whence it follows that $\mu(A_n) \to 0$ as $n \to \infty$.

('if') Let ν_n be the indefinite integral of $|\,h_n\,|$. Because the sequence $(|\,h_n\,|)$ is mean-fundamental, (ν_n) is a uniformly fundamental sequence of measures by 4.5, so that $\nu = \lim_n \nu_n$ is a measure.

We shall show that

$$\nu(A) = \lim_n \nu_n(A) = \lim_n \int_A |\,h_n\,|\; d\mu = 0 \tag{14}$$

for all $A \in \mathcal{S}$ such that $\mu(A) < \infty$.

Given $\varepsilon > 0$, let A_n be the set defined by (13) for each $n = 1, 2, \cdots$. Decompose

$$\int_A |\, h_n \,|\, d\mu = \int_{A \cap A_n} |\, h_n \,|\, d\mu + \int_{A \cap A_n{}'} |\, h_n \,|\, d\mu.$$

It is known that (ν_n) is equally absolutely continuous, i.e., there exists $\delta > 0$ such that $\nu_n(B) < \varepsilon$ for all n if $\mu(B) < \delta$. But, the assumption that $h_n \to 0$ in measure implies that there exists n_0 such that $\mu\,(A_n) < \delta$ if $n \geqslant n_0$. It follows that (the first term) $= \nu_n(A \cap A_n) < \varepsilon$ if $n \geqslant n_0$.

On the other hand, it is obvious that (the second term) $\leqslant \varepsilon\mu(A) < \infty$. Therefore

$$\int_A |\, h_n \,|\, d\mu \leqslant (1 + \mu(A))\varepsilon. \quad (n \geqslant n_0),$$

which proves (14).

To complete our proof, let A be a σ-finite support for all h_n, $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{S}$, $A_i \perp$ and $\mu(A_i) < \infty$.

Because $\nu(A_i) = 0$ for all i by (14), we have $\nu(A) = \sum_i \nu(A_i) = 0$. Here, we use the fact that $\nu = \lim_n \nu_n$ is a measure. Hence

$$\lim_n \int |\, h_n \,|\, d\mu = \lim_n \int_A |\, h_n \,|\, d\mu = 0,$$

which completes our proof. ||

Note. In the theorem above, mean fundamentality of (h_n) is essential. For, there exists a sequence (h_n) which converges in measure, but does not mean converge to 0: e.g. let $X = \mathbb{R}$, μ be Lebesgue measure, $h_n = \frac{1}{n}\chi_{[\,0\,,\,n\,]}$ for each $n = 1, 2, \cdots$. Then the sequence (h_n) uniformly converges (hence in measure) to 0, but $\int h_n \, d\mu = 1 \neq 0$ for all n.

4.7 Theorem. Let (f_n) be a mean fundamental sequence of integrable functions in $\mathcal{F}_E$. Then (f_n) mean converges to some integrable function f in $\mathcal{F}_E$.

Proof Because mean fundamentality implies fundamentality in measure, (f_n) converges to some function $f \in \mathcal{F}_E$ in measure. Then, the sequence $(f - f_n)$ in $\mathcal{F}_E$ is mean fundamental, converging in measure to 0, so that it mean converges to 0 by the previous theorem. Thus (f_n) mean converges to f. The integrability of f is obvious. ||

References

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